

SOME FORMULAS OF SANTALÓ TYPE IN FINSLER GEOMETRY AND ITS APPLICATIONS

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ABSTRACT. In this paper, we establish two Santaló type formulas for general Finsler manifolds. As applications, we derive a universal lower bound for the first eigenvalue of the nonlinear Laplacian, two Croke type isoperimetric inequalities, and a Yamaguchi type finiteness theorem in Finsler geometry.

1. INTRODUCTION

In [16, 17], Santaló considered the kinematic measure and established a formula which describes the Liouville measure on the unit sphere bundle of a Riemannian manifold in terms of the geodesic flow and the measure of a hypersurface. This formula plays an important role in global Riemannian geometry. Some of its applications are universal bounds for the first eigenvalue [5], Croke's isoperimetric inequality [10] and a generalization of Berger's theorem [8]. Moreover, with Santaló's formula, Croke in [7] solved a famous isoperimetric problem in dimension 4. See [5, 7, 8, 9, 10, 11, 16, 17] for more details.

A Finsler manifold is a differentiable manifold, on which every tangent space is endowed a Minkowski norm instead of a Euclidean norm. There is only one reasonable notion of the measure for Riemannian manifolds (cf. [4]). However, the measure on a Finsler manifold can be defined in various ways and essentially different results may be obtained, e.g., [1, 2, 18]. Hence, it is interesting to ask whether an analogue of Santaló's formula still holds for Finsler manifolds.

Let (M, F) be a Finsler manifold. Denote by $\pi : SM \rightarrow M$ the unit sphere bundle. If $F(y) = F(-y)$ for any $y \in SM$, then F is reversible. In a reversible Finsler manifold, the reverse of a geodesic is still a geodesic (see [3, 18]). In [23], Shen and Zhao considered the problem above and established a Santaló type formula for reversible Finsler manifolds.

There are infinitely many nonreversible Finsler metrics. For example, a Randers metric in the form $F = \alpha + \beta$ is non-reversible, where α is a Riemannian metric and β is a 1-form. The reverse of a geodesic in a non-reversible Finsler manifold is in general not a geodesic. Moreover, in a non-reversible Finsler manifold, the measure of a hypersurface induced by the inward normal vector field may be different from the one induced by the outward normal vector field (see Example 1 in Section 5 below). The purpose of this paper is to establish some Santaló type formulas for general Finsler manifolds.

Let $(M, \partial M, F, d\mu)$ be a compact Finsler manifold with smooth boundary, where F is possibly non-reversible and $d\mu$ is a measure on M . Denote by \mathbf{n}_+ and \mathbf{n}_- the

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unit inward and outward normal vector fields along ∂M , respectively. The measures on ∂M induced by \mathbf{n}_\pm are defined by $dA_\pm := i^*(\mathbf{n}_\pm]d\mu)$. Let $S^+\partial M$ and $S^-\partial M$ be the bundles of inwardly and outwardly pointing unit vectors along ∂M , i.e., $S^\pm\partial M = \{y \in SM|_{\partial M} : g_{\mathbf{n}_\pm}(\mathbf{n}_\pm, y) > 0\}$. The measures on $S^\pm\partial M$ are the product measures $d\chi_\pm(y) := d\nu_{\pi(y)}(y)dA_\pm(\pi(y))$, where $d\nu_x(y)$ is the Riemannian measure on $S_xM := \pi^{-1}(x)$ induced by F . For each $y \in S^+\partial M$, set $\mathfrak{t}(y) := \sup\{t > 0 : \gamma_y(s) \in M, 0 < s < t\}$ and $l(y) := \min\{i(y), \mathfrak{t}(y)\}$, where $i(y)$ is the cut value of y .

Since F may be non-reversible, to investigate the asymmetry of the Finsler manifold, we introduce the reverse of F , which is defined by $\tilde{F}(y) := F(-y)$. Clearly, \tilde{F} is a Finsler metric as well. Let $\tilde{\mathfrak{t}}(\cdot)$, $\tilde{i}(\cdot)$ and $\tilde{l}(\cdot)$ be defined as above on $(M, \partial M, \tilde{F})$. Then we have the following Santaló type formulas.

Theorem 1.1. *For all integral function f on SM , we have*

$$\begin{aligned} \text{(i)} \quad & \int_{\mathcal{V}_M^-} f dV_{SM} = \int_{y \in S^+\partial M} e^{\tau(y)} g_{\mathbf{n}_+}(\mathbf{n}_+, y) d\chi_+(y) \int_0^{l(y)} f(\varphi_t(y)) dt, \\ \text{(ii)} \quad & \int_{\mathcal{V}_M^+} f dV_{SM} = \int_{y \in S^-\partial M} e^{\tau(y)} g_{\mathbf{n}_-}(\mathbf{n}_-, y) d\chi_-(y) \int_0^{\tilde{l}(-y)} f(\varphi_{-t}(y)) dt, \end{aligned}$$

where dV_{SM} is the canonical Riemannian measure on SM , τ is the distortion of $d\mu$, $\varphi_t(y)$ is the geodesic flow of F , $\mathcal{V}_M^- := \{y \in SM : \tilde{\mathfrak{t}}(-y) \leq \tilde{i}(-y)\}$ and $\mathcal{V}_M^+ := \{y \in SM : \mathfrak{t}(y) \leq i(y)\}$.

One can easily see that Theorem 1.1 implies the Santaló type formulas for reversible Finsler manifolds [23] and for Riemannian manifolds [16, 17]. It is remarkable that, in a non-reversible Finsler manifold, $A_-(\partial M) \neq A_+(\partial M)$ and the formulas (1) and (2) contain information about $A_+(\partial M)$ and $A_-(\partial M)$, respectively.

Before giving some applications of Theorem 1.1, we shall recall some notions and basic facts of the first eigenvalue in the Finsler setting. The first eigenvalue $\lambda_1(M, d\mu)$ in $(M, F, d\mu)$ is defined as the smallest positive eigenvalue of the nonlinear Laplacian $\Delta_{d\mu}$ introduced by Shen (cf. [14, 18, 19]). It should be noted that both $\Delta_{d\mu}$ and $\lambda_1(M, d\mu)$ are dependent on the choice of the measure $d\mu$. Theorem 1.1 now yields the following

Theorem 1.2. *Let $(M, \partial M, F)$ be a compact Finsler n -manifold with smooth boundary such that every geodesic ray in (M, F) minimizes distance up to the point that it intersects ∂M . Then*

$$\lambda_1(M, d\mu) \geq \begin{cases} \frac{\lambda_D(\mathbb{S}_D^+)}{\Lambda_F^{2n+1}}, & d\mu \text{ is the Busemann-Hausdorff measure,} \\ \frac{\lambda_D(\mathbb{S}_D^+)}{\Lambda_F^{2n+1}}, & d\mu \text{ is the Holmes-Thompson measure,} \end{cases}$$

where $D := \text{diam}(M)$, Λ_F is the uniform constant of F , and \mathbb{S}_D^+ denotes the n -dimensional Riemannian hemisphere of the constant sectional curvature sphere having diameter equal to D . The equality holds if and only if (M, F) is isometric to \mathbb{S}_D^+ .

Note that a Finsler metric F is Riemannian if and only if $\Lambda_F = 1$. Hence, Theorem implies Croke's sharp universal lower bound for the first eigenvalue [5, 10].

Let $(M, \partial M, F)$ be as before. For each $x \in M$, set

$$\omega := \inf_{x \in M} \min\{\omega_x^+, \omega_x^-\},$$

where $\omega_x^\pm := c_{n-1}^{-1} \int_{U_x^\pm} e^{\tau(y)} d\nu_x(y)$, $U_x^\pm := \pi|_{V_M^\pm}^{-1}(x)$ and $c_{n-1} = \text{Vol}(\mathbb{S}^{n-1})$. Then Theorem 1.1 furnishes the following inequalities.

Theorem 1.3. *Let $(M, \partial M, F, d\mu)$ be a compact Finsler n -manifold with smooth boundary, where $d\mu$ is either the Busemann-Hausdorff measure or the Holmes-Thompson measure. Then*

(1)

$$\frac{A_\pm(\partial M)}{\mu(M)} \geq \frac{(n-1)c_{n-1}\omega}{c_{n-2} D \Lambda_F^{2n+\frac{1}{2}}},$$

where $D := \text{diam}(M)$.

(2)

$$\frac{A_\pm(\partial M)}{\mu(M)^{1-\frac{1}{n}}} \geq \frac{c_{n-1}\omega^{1+\frac{1}{n}}}{(c_n/2)^{1-\frac{1}{n}} \Lambda_F^{2n+\frac{5}{2}}},$$

with equality if and only if (M, F) is a Riemannian hemisphere of a constant sectional curvature sphere.

If F is reversible, then $\omega_+ = \omega_-$ and $A_+(\partial M) = A_-(\partial M)$. Hence, Theorem 1.3 implies Croke type isoperimetric inequalities for reversible Finsler manifolds [23, Theorem 1.6] and for Riemannian manifolds [10].

As an application of Theorem 1.3, we obtain a Finslerian version of Yamaguchi's finiteness theorem.

Theorem 1.4. *For any n and positive numbers i, V, δ , the class of closed Finsler n -manifolds (M, F) with injectivity radius $i_M \geq i$, $\Lambda_F \leq \delta$ and $\mu(M) \leq V$, contains at most finitely many homotopy types. Here, $\mu(M)$ is either the Busemann-Hausdorff volume or the Holmes-Thompson volume of M .*

2. PRELIMINARIES

In this section, we recall some definitions and properties about Finsler manifolds. See [3, 18] for more details.

Let (M, F) be a (connected) Finsler n -manifold with Finsler metric $F : TM \rightarrow [0, \infty)$. Let $(x, y) = (x^i, y^i)$ be local coordinates on TM . Define

$$g_{ij}(x, y) := \frac{1}{2} \frac{\partial^2 F^2(x, y)}{\partial y^i \partial y^j}, \quad G^i(y) := \frac{1}{4} g^{il}(y) \left\{ 2 \frac{\partial g_{jl}}{\partial x^k}(y) - \frac{\partial g_{jk}}{\partial x^l}(y) \right\} y^j y^k,$$

where G^i are the geodesic coefficients. A smooth curve $\gamma(t)$ in M is called a (constant speed) geodesic if it satisfies

$$\frac{d^2 \gamma^i}{dt^2} + 2G^i \left(\frac{d\gamma}{dt} \right) = 0.$$

We always use $\gamma_y(t)$ to denote the geodesic with $\dot{\gamma}_y(0) = y$.

The Ricci curvature is defined by $\mathbf{Ric}(y) := \sum_{i=1}^n R_i^i(y)$, where

$$R_k^i(y) := 2 \frac{\partial G^i}{\partial x^k} - y^j \frac{\partial^2 G^i}{\partial x^j \partial y^k} + 2G^j \frac{\partial^2 G^i}{\partial y^j \partial y^k} - \frac{\partial G^i}{\partial y^j} \frac{\partial G^j}{\partial y^k}.$$

Let $\pi : SM \rightarrow M$ be the unit sphere bundle, i.e., $S_x M := \{y \in T_x M : F(x, y) = 1\}$ and $SM := \cup_{x \in M} S_x M$. The measure on SM is defined by

$$\begin{aligned} dV_{SM}|_{(x,y)} &= \sqrt{\det g_{ij}(x, y)} dx^1 \wedge \cdots \wedge dx^n \wedge d\nu_x(y) \\ &= e^{\tau(y)} \pi^*(d\mu(x)) \wedge d\nu_x(y). \end{aligned}$$

where

$$d\nu_x(y) := \sqrt{\det g_{ij}(x, y)} \left(\sum_{i=1}^n (-1)^{i-1} y^i dy^1 \wedge \cdots \wedge \widehat{dy^i} \wedge \cdots \wedge dy^n \right).$$

is the Riemannian measure on $S_x M$ induced by F .

The reversibility λ_F and the uniformity constant Λ_F of (M, F) are defined by $\lambda_F := \sup_{x \in M} \lambda_F(x)$ and $\Lambda_F := \sup_{x \in M} \Lambda_F(x)$, where

$$\lambda_F(x) := \sup_{y \in S_x M} F(x, -y), \quad \Lambda_F(x) := \sup_{X, Y, Z \in S_x M} \frac{g_X(Y, Y)}{g_Z(Y, Y)}.$$

Clearly, $\Lambda_F \geq \lambda_F^2 \geq 1$. $\lambda_F = 1$ if and only if F is reversible, while $\Lambda_F = 1$ if and only if F is Riemannian.

The dual Finsler metric F^* on M is defined by

$$F^*(\eta) := \sup_{X \in T_x M \setminus 0} \frac{\eta(X)}{F(X)}, \quad \forall \eta \in T_x^* M.$$

The Legendre transformation $\mathfrak{L} : TM \rightarrow T^*M$ is defined as

$$\mathfrak{L}(X) := \begin{cases} g_X(X, \cdot) & X \neq 0, \\ 0 & X = 0. \end{cases}$$

In particular, $F^*(\mathfrak{L}(X)) = F(X)$. Now let $f : M \rightarrow \mathbb{R}$ be a smooth function on M . The gradient of f is defined by $\nabla f = \mathfrak{L}^{-1}(df)$. Thus, $df(X) = g_{\nabla f}(\nabla f, X)$.

Let $d\mu$ be a measure on M . In a local coordinate system (x^i) , express $d\mu = \sigma(x) dx^1 \wedge \cdots \wedge dx^n$. In particular, the Busemann-Hausdorff measure $d\mu_{BH}$ and the Holmes-Thompson measure $d\mu_{HT}$ are defined by

$$\begin{aligned} d\mu_{BH} &= \sigma_{BH}(x) dx := \frac{\text{Vol}(\mathbb{B}^n)}{\text{Vol}(\{y \in T_x M : F(x, y) < 1\})} dx^1 \wedge \cdots \wedge dx^n, \\ d\mu_{HT} &= \sigma_{HT}(x) dx := \left(\frac{1}{c_{n-1}} \int_{S_x M} \sqrt{\det g_{ij}(x, y)} d\nu_x(y) \right) dx^1 \wedge \cdots \wedge dx^n. \end{aligned}$$

For $y \in T_x M \setminus 0$, define the distortion of $(M, F, d\mu)$ as

$$\tau(y) := \log \frac{\sqrt{\det g_{ij}(x, y)}}{\sigma(x)}.$$

By the same argument as in [21], one can show the following lemma.

Lemma 2.1. *Let (M, F) be a Finsler n -manifold with finite uniform constant Λ_F . Let $d\mu$ denote either the Busemann-Hausdorff measure or the Holmes-Thompson measure on M . Then the distortion τ of $d\mu$ satisfy $\Lambda_F^{-n} \leq e^{\tau(y)} \leq \Lambda_F^n$, for all $y \in SM$.*

The reverse of a Finsler metric F is defined by $\tilde{F}(y) := F(-y)$. It is not hard to see that $\tilde{G}^i(y) = G^i(-y)$ and $d\tilde{\mu} = d\mu$, where \tilde{G}^i (resp. G^i) are the geodesic coefficients of \tilde{F} (resp. F), and $d\tilde{\mu}$ (resp. $d\mu$) denotes the Busemann-Hausdorff

measure or the Holmes-Thompson measure of \tilde{F} (resp. F). In particular, if γ is a geodesic of F , then the reverse of γ is a geodesic of \tilde{F} .

3. SANTALÓ TYPE FORMULAS

Let $(M, \partial M, F)$ be compact Finsler manifold with smooth boundary. Denote by \mathbf{n}_+ (resp. \mathbf{n}_-) the unit inward (resp. outward) normal vector field along ∂M . Define $\mathcal{N}_+ := \{k \cdot \mathbf{n}_+(x) : x \in \partial M, k \in \mathbb{R}\}$. The exponential map Exp_+ of \mathcal{N}_+ is defined by

$$\text{Exp}_+ : \mathcal{N}_+ \rightarrow M, k \cdot \mathbf{n}_+(x) \mapsto \exp_x(k\mathbf{n}_+(x)).$$

We always identify ∂M with the zero section of \mathcal{N}_+ . The same arguments as in [23, Lemma 5.1, Remark 5.1] show the following lemma.

Lemma 3.1. *Exp_+ maps a neighborhood of $\partial M \subset \mathcal{N}_+$ C^1 -diffeomorphically onto a neighborhood of $\partial M \subset \overline{M}$. Hence, there exists a small $\delta > 0$ such that $\text{Exp}_+ : M_\delta \rightarrow \text{Exp}_+(M_\delta)$ is C^1 -diffeomorphic, where $M_\delta := \{k \cdot \mathbf{n}_+(x) : x \in \partial M, 0 \leq k < \delta\}$.*

Define $\rho : \overline{M} \rightarrow \mathbb{R}_+$ by $\rho(x) = d(\partial M, x)$. Lemma 3.1 together with the proofs of [23, Lemma 5.2-5.3, Corollary 5.1] and [18, Lemma 3.2.3] yields

Lemma 3.2. *Let $\sigma(t)$, $0 \leq t < \epsilon$, be a C^1 -curve with $\sigma(0) \in \partial M$ and $\sigma((0, \epsilon)) \subset M$. Then*

$$0 \leq \frac{d}{dt} \Big|_{t=0^+} \rho \circ \sigma(t) = g_{\mathbf{n}_+}(\mathbf{n}_+, \dot{\sigma}(0)).$$

Hence, $\nabla \rho_+(x) = \mathbf{n}_+(x)$, for any $x \in \partial M$.

Set $S^\pm \partial M := \{y \in SM|_{\partial M} : g_{\mathbf{n}_\pm}(\mathbf{n}_\pm, y) > 0\}$. By the Legendre transformations, one can show that $S^\pm \partial M$ are two submanifolds of \overline{SM} .

Remark 1. In general, $\mathbf{n}_+ \neq -\mathbf{n}_-$. However, it follows from the Legendre transformations that $S^\pm \partial M = \{y \in SM|_{\partial M} : g_{\mathbf{n}_\mp}(\mathbf{n}_\mp, y) < 0\}$.

Set $\mathcal{Z} := \{y \in S\partial M : \exists t > 0 \text{ such that } \gamma_y((0, t)) \subset M\}$. Define a function $\mathbf{t} : SM \cup S^+ \partial M \cup \mathcal{Z} \rightarrow \mathbb{R}_+$ by $\mathbf{t}(y) := \sup\{t > 0 : \gamma_y(s) \in M, 0 < s < t\}$, which is called the \mathbf{t} -function. By the same argument as in [23, Lemma 5.4], one can show that \mathbf{t} -function is low semi-continuous on $SM \cup S^+ \partial M$.

Since $(M, \partial M, F)$ is compact, we can define a map

$$\Psi : \{(t, y) : y \in S^+ \partial M, 0 \leq t \leq \mathbf{t}(y)\} \rightarrow SM, (t, y) \mapsto \varphi_t(y),$$

where φ_t is the geodesic flow of F . Let $\tilde{\mathbf{t}}$ (resp. \tilde{i}) denote the \mathbf{t} -function (resp. the cut value function) defined on $(M, \partial M, \tilde{F})$, where $\tilde{F}(y) := F(-y)$. Set

$$U_M^- := \{y \in SM : \tilde{\mathbf{t}}(-y) < \tilde{i}(-y)\}.$$

Since $y \in SM$ implies that $\tilde{F}(-y) = 1$, U_M^- is well-defined. In particular, we have the following

Lemma 3.3. *$\Psi|_{\mathfrak{N}_+} : \mathfrak{N}_+ \rightarrow U_M^- \setminus U_{\mathcal{Z}}$ is a one-one map. Here, $\mathfrak{N}_+ := \{(t, y) : y \in S^+ \partial M, t \in (0, l(y))\}$, $U_{\mathcal{Z}} := \{\varphi_t(y) : y \in \mathcal{Z}, t \in (0, l(y))\}$, and $l(y) := \min\{i(y), \mathbf{t}(y)\}$.*

Proof. Since \overline{M} is compact, for each $y \in U_M^-$, $0 < \tilde{\mathbf{t}}(-y) < \tilde{i}(-y) < \infty$. Clearly, $\tilde{\gamma}_{-y}(t)$, $0 \leq t \leq \tilde{\mathbf{t}}(-y)$ is a unit speed minimal geodesic in $(\overline{M}, \tilde{F})$. Set $Y := -\tilde{\gamma}_{-y}(\tilde{\mathbf{t}}(-y))$. Thus,

$$F(Y) = \tilde{F}(-Y) = \tilde{F}(\dot{\tilde{\gamma}}_{-y}(\tilde{\mathbf{t}}(-y))) = 1.$$

It follows from Lemma 3.2 that $g_{\mathbf{n}_+}(\mathbf{n}_+, Y) \geq 0$. Hence, $Y \in S^+ \partial M \cup \mathcal{Z}$.

Let d (resp. \tilde{d}) denote the distance function induced by F (resp. \tilde{F}). Let $p := \pi(y)$ and $q := \pi(Y)$. Then $L_F(\gamma_Y([0, \tilde{\mathbf{t}}(-y)])) = \tilde{\mathbf{t}}(-y) = \tilde{d}(p, q) = d(q, p)$, which implies that $i(Y) \geq \tilde{\mathbf{t}}(-y)$. We claim that $i(Y) > \tilde{\mathbf{t}}(-y)$. If not, then p is the cut point of q along γ_Y . If p is also a conjugate point of q , then there exists a non-vanishing Jacobi field $J(t)$ along $\gamma_Y(t)$ such that $J(0) = 0$ and $J(\tilde{\mathbf{t}}(-y)) = 0$. It is easy to check that $\tilde{J}(t) := J(\tilde{\mathbf{t}}(-y) - t)$ is a Jacobi field along $\tilde{\gamma}_{-y}$ in $(\overline{M}, \tilde{F})$. Hence, q is a conjugate point of p along $\tilde{\gamma}_{-y}$ in $(\overline{M}, \tilde{F})$, which contradicts $\tilde{\mathbf{t}}(-y) < \tilde{i}(-y)$. Since p is not a conjugate point of q , by the proof of [3, Proposition 8.2.1], one can show that there exists another minimal geodesic from q to p in (\overline{M}, F) . Thus, there exist two distinct minimal geodesic from p to q with the length $\tilde{\mathbf{t}}(-y)$ in $(\overline{M}, \tilde{F})$, which also contradicts $\tilde{\mathbf{t}}(-y) < \tilde{i}(-y)$. Hence, the claim is true, which implies that $\tilde{\mathbf{t}}(-y) < \min\{\mathbf{t}(Y), i(Y)\} = l(Y)$.

From above, we show that for each $y \in U_M^-$, there exist $Y \in S^+ \partial M \cup \mathcal{Z}$ and $t := \tilde{\mathbf{t}}(-y) < l(Y)$ such that $y = \Psi(t, Y)$. Let $N_{\mathcal{Z}} := \{(t, y) : y \in \mathcal{Z}, t \in (0, l(y))\}$. Then $\Psi|_{\mathfrak{N}_+ \cup N_{\mathcal{Z}}} : \mathfrak{N}_+ \cup N_{\mathcal{Z}} \rightarrow U_M^-$ is surjective. Since Ψ is injective, we are done by $\Psi(N_{\mathcal{Z}}) = U_{\mathcal{Z}}$. \square

Given any measure $d\mu$ on M , the induced volume forms on ∂M by \mathbf{n}_{\pm} are defined by $dA_{\pm} := i^*(\mathbf{n}_{\pm} \lrcorner d\mu)$, where $i : \partial M \hookrightarrow M$ is the inclusion map (cf. [18]). Now we have the following Santaló type formulas.

Theorem 3.4. *Let $(M, \partial M, F, d\mu)$ be a compact Finsler manifold with smooth boundary. Thus, for all integral function f on SM , we have*

$$\begin{aligned} (1) \quad \int_{\mathcal{V}_M^-} f dV_{SM} &= \int_{y \in S^+ \partial M} e^{\tau(y)} g_{\mathbf{n}_+}(\mathbf{n}_+, y) d\chi_+(y) \int_0^{l(y)} f(\varphi_t(y)) dt, \\ (2) \quad \int_{\mathcal{V}_M^+} f dV_{SM} &= \int_{y \in S^- \partial M} e^{\tau(y)} g_{\mathbf{n}_-}(\mathbf{n}_-, y) d\chi_-(y) \int_0^{\tilde{l}(-y)} f(\varphi_{-t}(y)) dt, \end{aligned}$$

where $\mathcal{V}_M^- := \{y \in SM : \tilde{\mathbf{t}}(-y) \leq \tilde{i}(-y)\}$, $\mathcal{V}_M^+ := \{y \in SM : \mathbf{t}(y) \leq i(y)\}$ and $d\chi_{\pm}(y) = dA_{\pm}(\pi(y)) \wedge d\nu_{\pi(y)}(y)$.

Proof. (1). Given any $y \in S^+ \partial M$. We identify $T_y(S^+ \partial M)$ with its image in $T_{(0,y)}(\mathbb{R} \times S^+ \partial M)$. Since $\Psi_{*(0,y)}(X) = X$, $\forall X \in T_y(S^+ \partial M)$, we have

$$(3.1) \quad \Psi^*(d\chi_+(y)) \equiv d\chi_+|_{(0,y)} \pmod{dt}.$$

We claim that $[\Psi^* \pi^* d\rho]|_{(0,y)} \equiv 0 \pmod{dt}$. In fact, for each $X \in T_y(S^+ \partial M)$, there exists a curve $\xi : [0, +\varepsilon) \rightarrow S^+ \partial M$ with $\xi(0) = y$ and $\dot{\xi}(0) = X$. Thus,

$$\langle X, \Psi^* \pi^* d\rho \rangle|_{(0,y)} = \langle \pi_* (\Psi_{*(0,y)} X), d\rho \rangle = \langle \pi_* X, d\rho \rangle = \left. \frac{d}{ds} \right|_{s=0} \rho(\pi(\xi(s))) = 0.$$

The claim is true. Lemma 3.2 now yields

$$\begin{aligned}
 [\Psi^* \pi_1^* d\rho]|_{(0,y)} &= \left\langle \frac{\partial}{\partial t}, \Psi^* \pi_1^* d\rho \right\rangle_{(0,y)} dt \\
 (3.2) \quad &= \left(\frac{d}{dt} \Big|_{t=0^+} \rho \circ \gamma_y(t) \right) dt = g_{\mathbf{n}_+}(\mathbf{n}_+, y) dt.
 \end{aligned}$$

Define a function $\eta \in C^\infty(\mathbb{R} \times S^+ \partial M)$ by $\Psi^*(dV_{SM}) = \eta \cdot \beta$, where $\beta|_{(t,y)} = dt \wedge d\chi_+(y)$ is a $(2n-1)$ form on $\mathbb{R} \times S^+ \partial M$. It is easy to check that $\eta(t, y) = \eta(0, y)$ (cf. [23, Lemma 5.6]). By the co-area formula (see [18, Theorem 3.3.1]), (3.1) and (3.2), we have

$$\begin{aligned}
 [\eta dt \wedge d\chi_+]|_{(0,y)} &= \Psi^*(dV_{SM}(y)) = \Psi^*[e^{\tau(y)} \pi^*(d\mu)(y) \wedge d\nu_{\pi(y)}(y)] \\
 &= \Psi^*[e^{\tau(y)} \pi^*(d\rho \wedge dA_+)(y) \wedge d\nu_{\pi(y)}(y)] \\
 &= [e^{\tau(y)} g_{\mathbf{n}_+}(\mathbf{n}_+, y) dt \wedge d\chi_+]|_{(0,y)},
 \end{aligned}$$

that is, $\eta(0, y) = e^{\tau(y)} g_{\mathbf{n}_+}(\mathbf{n}_+, y)$. It follows from the definition of η that

$$(3.3) \quad \Psi^*(dV_{SM}(\varphi_t(y))) = e^{\tau(y)} g_{\mathbf{n}_+}(\mathbf{n}_+, y) dt \wedge d\chi,$$

which implies that Ψ is of maximal rank. Hence, Lemma 3.3 yields that $\Psi|_{\mathfrak{N}_+}$ is a diffeomorphism.

Let $\mathcal{N} := \{y \in SM : \tilde{\mathbf{t}}(-y) = \tilde{i}(-y)\}$. Thus, $\mathcal{V}_M^- = U_M^- \cup \mathcal{N}$. By an argument similar to the proof of Lemma 3.3, one has $\mathcal{N} \subset \{\varphi_{l(y)} y : y \in S^+ \partial M \cup \mathcal{Z}, l(y) = i(y)\}$, which implies that \mathcal{N} has measure zero with respect to dV_{SM} . Also note that $V_{SM}(U_M^- \setminus \Psi(\mathfrak{N}_+)) = V_{SM}(U_{\mathcal{Z}}) = 0$. Hence, by (3.3), we have

$$\begin{aligned}
 \int_{\mathcal{V}_M^-} f dV_{SM} &= \int_{U_M^-} f dV_{SM} \\
 &= \int_{\Psi(\mathfrak{N}_+)} f dV_{SM} = \int_{\mathfrak{N}_+} \Psi^*(f dV_{SM}) \\
 &= \int_{S^+ \partial M} e^{\tau(y)} g_{\mathbf{n}_+}(\mathbf{n}_+, y) d\chi(y) \int_0^{l(y)} f(\varphi_t(y)) dt.
 \end{aligned}$$

(2). By considering $(M, \partial M, \tilde{F})$ and using the formula (1), we have

$$\int_{y \in \widetilde{\mathcal{V}_M^-}} f(-y) d\widetilde{V_{SM}}(y) = \int_{y \in \widetilde{S^+ \partial M}} e^{\tilde{\tau}(y)} \tilde{g}_{\tilde{\mathbf{n}}_+}(\tilde{\mathbf{n}}_+, y) d\tilde{\chi}_+(y) \int_0^{\tilde{l}(y)} f(-\tilde{\varphi}_t(y)) dt,$$

where the quantities $\tilde{*}$ denote the quantities $*$ defined by \tilde{F} . Note that $\tilde{\mathbf{n}}_+ = -\mathbf{n}_-$ and $-\tilde{\varphi}_t(y) = \varphi_{-t}(-y)$, $0 \leq t \leq \tilde{l}(y)$. The formula (2) now follows from the transformation $y \mapsto -y$. \square

4. A UNIVERSAL LOWER BOUND FOR THE FIRST EIGENVALUE OF THE NONLINEAR LAPLACIAN

Definition 4.1 ([14, 19]). Let $(M, F, d\mu)$ be a compact Finsler manifold. Denote $\mathcal{H}_0(M, d\mu)$ by

$$\mathcal{H}_0(M, d\mu) := \begin{cases} \{f \in W_2^1(M) : \int_M f d\mu = 0\}, & \partial M = \emptyset, \\ \{f \in W_2^1(M) : f|_{\partial M} = 0\}, & \partial M \neq \emptyset. \end{cases}$$

Define the canonical energy functional $E_{d\mu}$ on $\mathcal{H}_0(M, d\mu) - \{0\}$ by

$$E_{d\mu}(u) := \frac{\int_M F^*(du)^2 d\mu}{\int_M u^2 d\mu}.$$

λ is an eigenvalue if there is a function $u \in \mathcal{H}_0(M, d\mu) - \{0\}$ such that $d_u E_{d\mu} = 0$ with $\lambda = E_{d\mu}(u)$. In this case, u is called an eigenfunction corresponding to λ . The first eigenvalue $\lambda_1(M, d\mu)$ is defined by

$$\lambda_1(M, d\mu) := \inf_{u \in \mathcal{H}_0(M, d\mu) - \{0\}} E_{d\mu}(u),$$

which is the smallest positive critical value of $E_{d\mu}$.

Remark 2. u is an eigenfunction corresponding to λ if and only if

$$\Delta_{d\mu} u + \lambda u = 0 \text{ (in the weak sense),}$$

where $\Delta_{d\mu}$ is the nonlinear Laplacian introduced by Shen [14, 18, 19]. It should be noted that $\Delta_{d\mu}$ is dependent on the choice of $d\mu$.

Proposition 4.2. *Let (M, F) be a Finsler n -manifold. Then for any $p \in M$ and $f \in C^\infty(M)$, we have*

$$(4.1) \quad F^*(df|_p)^2 \geq \frac{n}{c_{n-1}\Lambda_F^{n+1}(p)} \int_{S_p M} \langle y, df \rangle^2 d\nu_p(y),$$

with equality if and only if $F(p, \cdot)$ is a Euclidean norm.

Proof. Without loss of generality, we may suppose $df|_p \neq 0$. Set $B_p M := \{y \in T_p M : F(p, y) < 1\}$. By [21], one can choose a $g_{\nabla f}$ -orthonormal basis $\{e_i\}$ of $T_p M$ such that $e_n = \nabla f / F(\nabla f)$ and $\deg g_{ij}(p, y) \leq \Lambda_F^n(p)$. Let $\{y^i\}$ denote the corresponding coordinates. By Stokes' formula, we have

$$\begin{aligned} & \int_{S_p M} \langle y, df \rangle^2 d\nu_p(y) \\ & \leq \Lambda_F^{\frac{n}{2}}(p) F^2(\nabla f) \int_{S_p M} (y^n)^2 \sum_{k=1}^n (-1)^{k-1} y^k dy^1 \wedge \cdots \wedge \widehat{dy^k} \wedge \cdots \wedge dy^n \\ & = (n+2) \Lambda_F^{\frac{n}{2}}(p) F^2(\nabla f) \int_{B_p M} (y^n)^2 dy^1 \wedge \cdots \wedge dy^n \\ (4.2) \quad & \leq (n+2) \Lambda_F^{\frac{n}{2}}(p) F^2(\nabla f) \int_{\mathbb{B}^n(\sqrt{\Lambda_F(p)})} (y^n)^2 dy^1 \wedge \cdots \wedge dy^n \\ & = \frac{c_{n-1}}{n} \Lambda_F^{n+1}(p) F^2(\nabla f) \end{aligned}$$

If equality holds in (4.1), then it follows from (4.2) that $B_p M = \mathbb{B}^n(\sqrt{\Lambda_F(p)})$. Namely, $F(y) = 1$ if and only if $g_{\nabla f}(y, y) = \Lambda_F(p)$. In particular, $1 = F(e_n) = g_{\nabla f}(e_n, e_n) = \Lambda_F(p)$, which implies that $F(p, \cdot)$ is a Euclidean norm. \square

Theorem 4.3. *Let $(M, \partial M, F)$ be a compact Finsler n -manifold with smooth boundary such that every geodesic ray in (M, F) minimizes distance up to the point that it intersects ∂M . Then*

$$(4.3) \quad \lambda_1(M, d\mu) \geq \begin{cases} \frac{\lambda_1(\mathbb{S}_D^+)}{\Lambda_F^{4n+1}}, & d\mu = d\mu_{BH}, \\ \frac{\lambda_1(\mathbb{S}_F^+)}{\Lambda_F^{2n+1}}, & d\mu = d\mu_{HT}, \end{cases}$$

where $D := \text{diam}(M)$ and \mathbb{S}_D^+ denotes the n -dimensional Riemannian hemisphere of the constant sectional curvature sphere having diameter equal to D . The equality holds if and only if (M, F) is isometric to \mathbb{S}_D^+ .

Proof. Lemma 2.1 yields that

$$(4.4) \quad \int_{S_p M} e^{\tau(y)} d\nu_p(y) = c_{n-1} \frac{\sigma_{HT}(p)}{\sigma(p)} \geq \begin{cases} \frac{c_{n-1}}{\Lambda_F^{2n}}, & d\mu = d\mu_{BH}, \\ c_{n-1}, & d\mu = d\mu_{HT}. \end{cases}$$

Since $\mathcal{V}_M^+ = SM$, Theorem 3.4 together with Proposition 4.2 and (4.4) then yields

$$(4.5) \quad \begin{aligned} & \int_M F^{*2}(df) d\mu \\ & \geq \frac{n}{c_{n-1} \Lambda_F^{n+1}} \int_M d\mu(p) \int_{S_p M} \langle y, df \rangle^2 d\nu_p(y) \\ & = \frac{n}{c_{n-1} \Lambda_F^{n+1}} \int_{SM} e^{-\tau(y)} \langle y, df \rangle^2 dV_{SM}(y) \\ & = \frac{n}{c_{n-1} \Lambda_F^{n+1}} \int_{y \in S^- \partial M} e^{\tau(y)} g_{\mathbf{n}_-}(\mathbf{n}_-, y) d\chi_-(y) \int_{-\tilde{l}(-y)}^0 e^{-\tau(\varphi_t(y))} \langle \varphi_t(y), df \rangle^2 dt \\ & \geq \frac{n}{c_{n-1} \Lambda_F^{2n+1}} \int_{y \in S^- \partial M} e^{\tau(y)} g_{\mathbf{n}_-}(\mathbf{n}_-, y) d\chi_-(y) \int_{-\tilde{l}(-y)}^0 \left(\frac{d}{dt} f(\gamma_y(t)) \right)^2 dt \\ & \geq \frac{n}{c_{n-1} \Lambda_F^{2n+1}} \int_{y \in S^- \partial M} e^{\tau(y)} g_{\mathbf{n}_-}(\mathbf{n}_-, y) d\chi_-(y) \int_{-\tilde{l}(-y)}^0 \left(\frac{\pi}{\tilde{l}(-y)} \right)^2 f^2(\gamma_y(t)) dt \\ & \geq \frac{n}{c_{n-1} \Lambda_F^{2n+1}} \left(\frac{\pi}{D} \right)^2 \int_{SM} f^2(\pi(y)) dV_{SM}(y) \\ & \geq \begin{cases} \frac{\lambda_1(\mathbb{S}_D^+)}{\Lambda_F^{4n+1}} \int_M f^2 d\mu, & d\mu = d\mu_{BH}, \\ \frac{\lambda_1(\mathbb{S}_D^+)}{\Lambda_F^{2n+1}} \int_M f^2 d\mu, & d\mu = d\mu_{HT}. \end{cases} \end{aligned}$$

If we have equality in (4.3), then (4.5) together with Proposition 4.2 implies $\Lambda_F = 1$. Hence, (M, F) is a Riemannian manifold and $\lambda_1(M) = \lambda_1(\mathbb{S}_D^+)$. By the standard argument (see [5, p.131] or [10]), one can show that (M, F) is isometric to \mathbb{S}_D^+ . \square

In [19], Shen shows that the first eigenvalue of a forward metric ball is bounded from above by a constant depending only on the dimension and lower bounds on the Ricci curvature and the S-curvature. From Theorem 4.3, we obtain a lower bound for the first eigenvalue of a forward metric ball.

Corollary 4.4. *Let $(M, F, d\mu)$ be a forward complete Finsler n -manifold of injectivity radius i_M . For any $0 < r < i_M/(1 + \sqrt{\Lambda_F})$ and any $p \in M$, we have*

$$\lambda_1(B_p^+(r)) \geq \begin{cases} \frac{\lambda_1(\mathbb{S}_{2\sqrt{\Lambda_F}r}^+)}{\Lambda_F^{4n+1}}, & d\mu = d\mu_{BH}, \\ \frac{\lambda_1(\mathbb{S}_{2\sqrt{\Lambda_F}r}^+)}{\Lambda_F^{2n+1}}, & d\mu = d\mu_{HT}. \end{cases}$$

with equality if and only if $B_p^+(r)$ is isometric to \mathbb{S}_{2r}^+ .

5. CROKE TYPE ISOPERIMETRIC INEQUALITIES

In this section, we shall establish Theorem 1.3 and give some applications.

Lemma 5.1. *For each $x \in \partial M$, we have*

$$\int_{S_x^\sharp \partial M} g_{\mathbf{n}_\sharp}(\mathbf{n}_\sharp, y) e^{\tau(y)} d\nu_x(y) \leq \frac{c_{n-2}}{n-1} \Lambda_F^{2n+\frac{1}{2}}(x),$$

with equality if and only if $F(x, \cdot)$ is a Euclidean norm. Here, " \sharp " denotes either "+" or "-", and $S_x^\sharp \partial M := \{y \in S_x M : g_{\mathbf{n}_\sharp}(\mathbf{n}_\sharp, y) > 0\}$.

Proof. Suppose $\sharp = +$. By [21], one can choose a $g_{\mathbf{n}_+}$ -orthnormal basis $\{e_i\}$ of $T_x M$ such that $e_n = \mathbf{n}_+$ and $\det g_{ij}(x, y) \leq \Lambda_F^n(x)$. Let $\{y^i\}$ be the corresponding coordinates. Set $\|\cdot\| := \sqrt{g_{\mathbf{n}_+}(\cdot, \cdot)}$. Define

$$B_x^+ := \{y \in T_x M : F(y) < 1, y^n > 0\}, \quad B_{x,r}^+ := \{y \in T_x M : F(y) = 1, y^n = r\}$$

$$\mathbb{B}_{x,r}(s) := \{y \in T_x M : y^n = r, \|y^\alpha e_\alpha\| < s\}, \quad \varpi := g_{\mathbf{n}_+}(\mathbf{n}_+, y) e^{\tau(y)} d\nu_p(y).$$

For each $y \in B_x^+$, $y^n = g_{\mathbf{n}_+}(\mathbf{n}_+, y) \leq F(\mathbf{n}_+)F(y) \leq 1$. Stokes' formula together with Lemma 2.1 then yields

$$\begin{aligned} \int_{S_x^+ \partial M} \varpi &\leq \Lambda_F^{3n/2}(x) \int_{S_x^+ \partial M} y^n \sum_{k=1}^n (-1)^{k-1} y^k dy^1 \wedge \cdots \wedge \widehat{dy^k} \wedge \cdots \wedge dy^n \\ &= (n+1) \Lambda_F^{\frac{3n}{2}}(x) \int_{B_x^+} y^n dy^1 \wedge \cdots \wedge dy^n \\ &= (n+1) \Lambda_F^{\frac{3n}{2}}(x) \int_0^1 \text{Vol}(B_{x,y^n}^+) y^n dy^n \\ &\leq (n+1) \Lambda_F^{\frac{3n}{2}}(x) \int_0^{\sqrt{\Lambda_F(x)}} \text{Vol}\left(\mathbb{B}_{x,y^n}(\sqrt{\Lambda_F(x) - (y^n)^2})\right) y^n dy^n \\ &= \frac{c_{n-2}}{n-1} \Lambda_F^{2n+\frac{1}{2}}(x), \end{aligned}$$

with equality if and only if $\Lambda_F(x) = 1$, i.e., $F(x, \cdot)$ is a Euclidean norm.

Suppose $\sharp = -$. Note that $\Lambda_F(x) = \Lambda_{\bar{F}}(x)$. Using the same method as in Theorem 3.4, one can get the formula. \square

Given any point $x \in M$, let (r, y) denote the polar coordinates about x . Set $\mathcal{F}(r, y) = e^{\tau(\gamma_y(r))} \hat{\sigma}_x(r, y)$, where $d\mu|_{(r,y)} =: \hat{\sigma}_x(r, y) dr \wedge d\nu_x(y)$. Then we have the following inequality of Berger-Kazdan type [23, Theorem 1.3]

Lemma 5.2 ([23]). *Let (M, F) be a compact Finsler n -manifold. For each $y \in SM$ and $0 < t \leq l \leq i_y$, we have*

$$\int_0^l dr \int_0^{l-r} \mathcal{F}(t, \varphi_r(y)) dt \geq \frac{\pi c_n}{2c_{n-1}} \left(\frac{l}{\pi}\right)^{n+1},$$

with equality if and only if

$$R_{\dot{\gamma}_y(t)}(\cdot, \dot{\gamma}_y(t)) \dot{\gamma}_y(t) = \left(\frac{\pi}{l}\right)^2 \text{id}, \quad 0 \leq t \leq l,$$

where R is the (Riemannian) curvature tensor acting on $\dot{\gamma}_y(t)^\perp$.

Now we have the following theorem.

Theorem 5.3. *Let $(M, \partial M, F, d\mu)$ be a compact Finsler n -manifold with smooth boundary, where $d\mu$ is either the Busemann-Hausdorff measure or the Holmes-Thompson measure. Set*

$$\omega := \inf_{x \in M} \min\{\omega_x^+, \omega_x^-\} = \min \left\{ \inf_{x \in M} \omega_x^+, \inf_{x \in M} \omega_x^- \right\},$$

where $\omega_x^\pm := c_{n-1}^{-1} \int_{U_x^\pm} e^{\tau(y)} d\nu_x(y)$ and $U_x^\pm := \pi|_{\mathcal{V}_M^\pm}^{-1}(x)$. Then

(1)

$$\frac{A_\pm(\partial M)}{\mu(M)} \geq \frac{(n-1)c_{n-1}\omega}{c_{n-2}D\Lambda_F^{2n+\frac{1}{2}}},$$

where $D := \text{diam}(M)$.

(2)

$$(5.1) \quad \frac{A_\pm(\partial M)}{\mu(M)^{1-\frac{1}{n}}} \geq \frac{c_{n-1}\omega^{1+\frac{1}{n}}}{(c_n/2)^{1-\frac{1}{n}}\Lambda_F^{2n+\frac{5}{2}}},$$

with equality if and only if (M, F) is a Riemannian hemisphere of a constant sectional curvature sphere.

Proof. (1) Theorem 1.1 together with Lemma 5.1 furnishes

$$\begin{aligned} c_{n-1}\omega\mu(M) &\leq c_{n-1} \int_M \omega_x^\mp d\mu(x) = \int_{x \in M} d\mu(x) \int_{U_x^\mp} e^{\tau(y)} d\nu_x(y) \\ &= V_{SM}(\mathcal{V}_M^\mp) \leq D \int_{S^\pm \partial M} e^{\tau(y)} g_{\mathbf{n}_\pm}(\mathbf{n}_\pm, y) d\chi_\pm(y) \\ &\leq D A_\pm(\partial M) \frac{c_{n-2}}{n-1} \Lambda_F^{2n+\frac{1}{2}}. \end{aligned}$$

(2) For each $y \in S^+ \partial M$, $l(\varphi_t(y)) \geq l(y) - t$, for any $0 \leq t \leq l(y)$. By Theorem 1.1, Lemma 2.1, Theorem 5.2 and Hölder's inequality, we have

$$\begin{aligned} &\mu^2(M) \\ &= \int_M d\mu(x) \int_{S_x M} d\nu_x(y) \int_0^{l(y)} \hat{\sigma}_x(r, y) dr = \int_{SM} dV_{SM}(y) \int_0^{l(y)} e^{-\tau(y)} \hat{\sigma}_x(r, y) dr \\ &\geq \int_{\mathcal{V}_M^-} dV_{SM}(y) \int_0^{l(y)} e^{-\tau(y)} \hat{\sigma}_x(r, y) dr \\ &= \int_{S^+ \partial M} e^{\tau(y)} g_{\mathbf{n}_+}(\mathbf{n}_+, y) d\chi_+(y) \int_0^{l(y)} dt \int_0^{l(\varphi_t(y))} e^{-\tau(\varphi_t(y))} \hat{\sigma}_x(r, \varphi_t(y)) dr \\ &\geq \Lambda_F^{-2n} \int_{S^+ \partial M} e^{\tau(y)} g_{\mathbf{n}_+}(\mathbf{n}_+, y) d\chi_+(y) \int_0^{l(y)} dt \int_0^{l(y)-t} \mathcal{F}(r, \varphi_t(y)) dr \\ &\geq \frac{c_n}{2c_{n-1}\pi^n \Lambda_F^{2n}} \int_{S^+ \partial M} l(y)^{n+1} e^{\tau(y)} g_{\mathbf{n}_+}(\mathbf{n}_+, y) d\chi_+(y) \\ &\geq \frac{c_n}{2c_{n-1}\pi^n \Lambda_F^{2n}} \left(\int_{S^+ \partial M} l(y) e^{\tau(y)} g_{\mathbf{n}_+}(\mathbf{n}_+, y) d\chi_+(y) \right)^{n+1} \left(\int_{S^+ \partial M} e^{\tau(y)} g_{\mathbf{n}_+}(\mathbf{n}_+, y) d\chi_+(y) \right)^{-n} \end{aligned}$$

$$\begin{aligned}
(5.2) \quad & \geq \frac{c_n}{2c_{n-1}\pi^n \Lambda_F^{2n}} V_{SM}(\mathcal{V}_M^-)^{n+1} \left(\frac{n-1}{c_{n-2} A_+(\partial M) \Lambda_F^{2n+\frac{1}{2}}} \right)^n \\
& \geq \frac{(c_{n-1})^n \omega^{n+1} \mu(M)^{n+1}}{(c_n/2)^{n-1} A_+^n(\partial M) \Lambda_F^{(2n+\frac{5}{2})n}}.
\end{aligned}$$

That is,

$$(5.3) \quad \frac{A_+(\partial M)}{\mu(M)^{1-\frac{1}{n}}} \geq \frac{c_{n-1} \omega^{1+\frac{1}{n}}}{(c_n/2)^{1-\frac{1}{n}} \Lambda_F^{2n+\frac{5}{2}}}.$$

Let \tilde{A}_\pm and $\tilde{\omega}$ be define as before on (M, \tilde{F}) . It is easy to check that $\tilde{A}_\pm(\partial M) = A_\mp(\partial M)$ and $\tilde{\omega} = \omega$. From above, we obtain

$$(5.4) \quad \frac{A_-(\partial M)}{\mu(M)^{1-\frac{1}{n}}} = \frac{\tilde{A}_+(\partial M)}{\tilde{\mu}(M)^{1-\frac{1}{n}}} \geq \frac{c_{n-1} \omega^{1+\frac{1}{n}}}{(c_n/2)^{1-\frac{1}{n}} \Lambda_F^{2n+\frac{5}{2}}}.$$

(5.3) together with (5.4) yields (5.1).

Suppose that equality holds in (5.1). Then we have equality in (5.3) or (5.4). It follows from (5.2) and Lemma 5.1 that $1 = \Lambda_F = \Lambda_{\tilde{F}}$. Hence, F is an Riemannian metric and (5.1) becomes

$$\frac{A(\partial M)}{\mu(M)^{1-\frac{1}{n}}} = \frac{c_{n-1} \omega^{1+\frac{1}{n}}}{(c_n/2)^{1-\frac{1}{n}}}.$$

Since $\mathcal{V}_M^- = SM$, $\mathfrak{t}(y) \leq i_y$, for all $y \in \overline{SM}$. Hölder's inequality implies $l(y)$ is constant, say, equal to l , on all of $S^+ \partial M$. Hence, $\mathfrak{t}(y) = l$, for all $y \in S^+ \partial M$. And Theorem 5.2 yields M has constant sectional curvature equal to $(\pi/l)^2$, i.e., M is a hemisphere. \square

From above, it is easy to see that Theorem 5.3 becomes Croke's isoperimetric inequality [10] if $\Lambda_F = 1$. In the Finslerian case, the upper bound on Λ_F in Theorem 5.3 is very important as the following example shows.

Example 1. Let \mathbb{B}^n be the unit open ball in \mathbb{R}^n equipped with a Funk metric F , that is,

$$F(x, y) = \frac{\sqrt{(1 - |x|^2)|y|^2 + (x \cdot y)^2} + x \cdot y}{1 - |x|^2},$$

where " $|\cdot|$ " (resp. " \cdot ") denotes the Euclidean norm (resp. inner product). For $r \in (0, 1)$, set $\Omega_r := \{x \in \mathbb{B}^n : |x| < r\}$. Then $(\Omega_r, \partial\Omega_r, F|_{\overline{\Omega_r}})$ is a compact Finsler manifold with smooth boundary. By directly computing, we have $\mu_{BH}(\Omega_r) = \frac{c_{n-1}}{n} r^n$ and $A_\pm(\partial\Omega_r) = c_{n-1}(1 \pm r)r^{n-1}$, where dA_\pm are induced by $d\mu_{BH}$. Clearly,

$$\lim_{r \rightarrow 1} \frac{A_+(\partial\Omega_r)}{A_-(\partial\Omega_r)} = +\infty.$$

Note that

$$\Lambda_{F|_{\overline{\Omega_r}}} = \left(\frac{1+r}{1-r} \right)^2, \quad \text{diam}(\Omega_r) = \log \left(\frac{1+r}{1-r} \right).$$

For any $x \in \Omega_r$,

$$\omega_x^\pm = \frac{1}{(1 - |x|^2)^{\frac{n+1}{2}}} \geq 1, \quad \text{i.e., } \omega = 1.$$

Hence, we have

$$\frac{A_{\pm}(\partial\Omega_r)}{\mu_{BH}(\Omega_r)} > \frac{(n-1)c_{n-1}\omega}{c_{n-2}\text{diam}(\Omega_r)\Lambda_{F|\overline{\Omega_r}}^{2n+\frac{1}{2}}}, \frac{A_{\pm}(\partial\Omega_r)}{\mu_{BH}(\Omega_r)^{1-\frac{1}{n}}} > \frac{c_{n-1}\omega^{1+\frac{1}{n}}}{(c_n/2)^{1-\frac{1}{n}}\Lambda_{F|\overline{\Omega_r}}^{2n+\frac{5}{2}}}.$$

In particular,

$$\lim_{r \rightarrow 1} \Lambda_{F|\overline{\Omega_r}} = +\infty, \lim_{r \rightarrow 1} \frac{A_{-}(\partial\Omega_r)}{\mu_{BH}(\Omega_r)} = \lim_{r \rightarrow 1} \frac{A_{-}(\partial\Omega_r)}{\mu_{BH}(\Omega_r)^{1-\frac{1}{n}}} = 0.$$

Before giving some applications of Theorem 5.3, we introduce the definitions of the Sobolev constant, Cheeger's constant and the isoperimetric constant of a closed Finsler manifold.

Definition 5.4. Let $(M, F, d\mu)$ be a closed Finsler manifold. The Sobolev constant $\mathcal{S}(M, d\mu)$ is defined as

$$\mathcal{S}(M, d\mu) := \inf_{f \in C^\infty(M)} \frac{\left\{ \int_M F^*(df) d\mu \right\}^n}{\inf_{\alpha \in \mathbb{R}} \left\{ \int_M |f - \alpha|^{\frac{n}{n-1}} d\mu \right\}^{n-1}}.$$

Cheeger's constant $\mathfrak{h}(M, d\mu)$ and the isoperimetric constant $\mathbb{I}(M, d\mu)$ are defined by

$$\mathfrak{h}(M, d\mu) := \inf_{\Gamma} \frac{\min\{A_{\pm}(\Gamma)\}}{\min\{\mu(M_1), \mu(M_2)\}}, \quad \mathbb{I}(M, d\mu) := \inf_{\Gamma} \frac{\min\{A_{\pm}(\Gamma)\}^n}{\{\min\{\mu(M_1), \mu(M_2)\}\}^{n-1}},$$

where Γ varies over compact $(n-1)$ -dimensional submanifolds of M which divide M into disjoint open submanifolds M_1, M_2 of M with common boundary $\partial M_1 = \partial M_2 = \Gamma$.

Remark 3. By using the co-area formula (cf. [18, Theorem 3.3.1]) and the same argument as in [13], one can obtain a Cheeger type inequality

$$\lambda_1(M, d\mu) \geq \frac{\mathfrak{h}^2(M, d\mu)}{4\lambda_F^2}.$$

And we also have a Federer-Fleming type inequality (see Proposition 6.1 below), i.e.,

$$\mathbb{I}(M, d\mu) \leq \mathcal{S}(M, d\mu) \leq 2\mathbb{I}(M, d\mu).$$

Corollary 5.5. Let $(M, F, d\mu)$ be a closed Finsler n -manifold with $\mathbf{Ric} \geq (n-1)k$, where $d\mu$ denotes either the Busemann-Hausdorff measure or the Holmes-Thompson measure. Then

$$\lambda_1(M, d\mu) \geq \left[\frac{(n-1)\mu(M)}{4c_{n-2}\Lambda_F^{4n+1}\text{diam}(M)\int_0^{\text{diam}(M)} \mathfrak{s}_k^{n-1}(t)dt} \right]^2,$$

$$\mathcal{S}(M, d\mu) \geq \frac{\mu(M)^{n+1}}{4c_{n-1}(c_n)^{n-1}\Lambda_F^{4n^2+\frac{9n}{2}}\left(\int_0^{\text{diam}(M)} \mathfrak{s}_k^{n-1}(t)dt\right)^{n+1}}.$$

Hence, both $\lambda_1(M, d\mu)$ and $\mathcal{S}(M, d\mu)$ can be bounded from below in terms of the diameter, volume, uniform constant and a lower bound for the Ricci curvature.

Proof. Step 1. Let Γ be any $(n-1)$ -dimensional compact submanifold of M dividing M into two open submanifolds M_1 and M_2 , such that $\partial M_1 = \partial M_2 = \Gamma$. Given $x \in M_1$, let

$$O_x := \{q \in M : \exists y \in U_x^- \text{ such that } q = \tilde{\gamma}_{-y}(t), t \in (0, \tilde{i}(-y))\},$$

where $\tilde{\gamma}_{-y}(t)$ is the geodesic in (M, \tilde{F}) with $\dot{\tilde{\gamma}}_{-y}(0) = -y$.

For any $q \in M_2$, there exists a minimal unit speed geodesic, say $\tilde{\gamma}_X(t)$, from x to q . Clearly, $\tilde{\gamma}_X(t)$ must hit the boundary and therefore, $\tilde{i}(X) \leq \tilde{i}(X)$. Since $F(-X) = \tilde{F}(X) = 1$, $q \in O_x$ which implies that $M_2 \subset O_x$.

Note that $\mathbf{Ric} \geq (n-1)k$, $\Lambda_{\tilde{F}} = \Lambda_F$ and $d\tilde{\mu} = d\mu$. Hence, by Lemma 2.1 and the volume comparison theorem (cf. [23, Theorem 3.1]), we have

$$\begin{aligned} \mu(M_2) &= \tilde{\mu}(M_2) \leq \tilde{\mu}(O_x) = \int_{y \in U_x^-} d\tilde{\nu}_x(-y) \int_0^{\tilde{i}(-y)} \tilde{\sigma}_x(r, -y) dr \\ &\leq \Lambda_F^n \int_{y \in U_x^-} d\tilde{\nu}_x(-y) \int_0^{\tilde{i}(-y)} \mathfrak{s}_k^{n-1}(r) dr \\ &\leq c_{n-1} \Lambda_F^{2n} \omega_1^-(x) \int_0^{\text{diam}(M)} \mathfrak{s}_k^{n-1}(r) dr. \end{aligned}$$

That is,

$$\omega_i^- \geq \frac{\mu(M_j)}{c_{n-1} \Lambda_F^{2n} \int_0^{\text{diam}(M)} \mathfrak{s}_k^{n-1}(r) dr}, \quad i \neq j.$$

Set $O'_x := \{q \in M : \exists y \in U_x^+ \text{ such that } q = \gamma_y(t), t \in (0, i(y))\}$. It is easy to see that $M_2 \subset O'_x$. By the similar argument, one can show that

$$\omega_i^+ \geq \frac{\mu(M_j)}{c_{n-1} \Lambda_F^{2n} \int_0^{\text{diam}(M)} \mathfrak{s}_k^{n-1}(t) dt}, \quad i \neq j.$$

Step 2. The inequalities above together with Theorem 5.3 yield

$$\begin{aligned} \mathfrak{h}(M, d\mu) &\geq \frac{(n-1)\mu(M)}{2c_{n-2} \Lambda_F^{4n+\frac{1}{2}} \text{diam}(M) \int_0^{\text{diam}(M)} \mathfrak{s}_k^{n-1}(t) dt}, \\ \mathfrak{I}(M, d\mu) &\geq \frac{\mu(M)^{n+1}}{4c_{n-1} (c_n)^{n-1} \Lambda_F^{4n^2+\frac{9n}{2}} \left(\int_0^{\text{diam}(M)} \mathfrak{s}_k^{n-1}(t) dt \right)^{n+1}}. \end{aligned}$$

Corollary now follows from Remark 3. \square

Corollary 5.6. *Let $(M, F, d\mu)$ be a closed Finsler n -manifold, where $d\mu$ is either the Busemann-Hausdorff measure or the Holmes-Thompson measure. Then for any $x \in M$ and $0 < r < i_M/(1 + \sqrt{\Lambda_F})$, we have*

$$\mu(B_x^+(r)) \geq \frac{C(n, \Lambda_F)}{n^n} r^n, \quad A_{\pm}(S_x^+(r)) \geq \frac{C(n, \Lambda_F)}{n^{n-1}} r^{n-1}.$$

Proof. The similar argument as in Lemma 3.3 shows $i_M = \tilde{i}_M$, where \tilde{i}_M is the injectivity radius of (M, \tilde{F}) . Hence, $U_x^{\pm} = S_x M$ for all $x \in B_x^+(r)$. By Theorem 5.3 and (4.4), we have

$$\frac{\frac{d}{dr} \mu(B_x^+(r))}{\mu(B_x^+(r))^{1-\frac{1}{n}}} = \frac{\Lambda_-(S_x^+(r))}{\mu(B_x^+(r))^{1-\frac{1}{n}}} \geq C(n, \Lambda_F),$$

which implies that

$$(5.5) \quad \mu(B_x^+(r)) \geq \frac{C(n, \Lambda_F)}{n^n} r^n.$$

Theorem 5.3 together with (5.5) yields

$$A_\pm(S_x^+(r)) \geq \frac{C(n, \Lambda_F)}{n^{n-1}} r^{n-1}.$$

□

In order to establish Theorem 1.5, let us recall some definitions and properties of general LGC spaces first. Refer to [20, 24] for more details.

Definition 5.7 ([20, 24]). A general metric space is a pair (X, d) , where X is a set and $d : X \times X \rightarrow \mathbb{R}^+ \cup \{\infty\}$, called a metric, is a function, satisfying the following two conditions: (a) $d(x, y) \geq 0$, with equality $\Leftrightarrow x = y$; (b) $d(x, y) + d(y, z) \geq d(x, z)$. The reversibility λ_X of a general metric space (X, d) is defined by $\lambda_X := \sup_{x \neq y} \frac{d(x, y)}{d(y, x)}$.

A contractibility function $\rho : [0, r) \rightarrow [0, +\infty)$ is a function satisfying: (a) $\rho(0) = 0$, (b) $\rho(\epsilon) \geq \epsilon$, (c) $\rho(\epsilon) \rightarrow 0$, as $\epsilon \rightarrow 0$, (d) ρ is nondecreasing. A general metric space X is LGC(ρ) for some contractibility function ρ , if for every $\epsilon \in [0, r)$ and $x \in X$, the forward ball $B_x^+(\epsilon)$ is contractible inside $B_x^+(\rho(\epsilon))$.

Lemma 5.8 ([24]). Fix a function $N : (0, \alpha) \rightarrow (0, \infty)$ with

$$\limsup_{\epsilon \rightarrow 0^+} \epsilon^n N(\epsilon) < \infty$$

and a contractibility function $\rho : [0, r) \rightarrow [0, \infty)$. The class

$$\mathcal{C}(N, \rho) := \{X \in \mathcal{M}^\delta : X \text{ is LGC}(\rho), \text{Cov}(X, \epsilon) \leq N(\epsilon) \text{ for all } \epsilon \in (0, \alpha)\}$$

contains only finitely many homotopy types. Here, \mathcal{M}^δ denotes the collection of compact general metric spaces with reversibility $\leq \delta$ and $\text{Cov}(X; \epsilon)$ denotes the minimum number of forward ϵ -balls it takes to cover X .

Corollary 5.6 together with Lemma 5.8 yields the following

Theorem 5.9. For any n and positive numbers i, V, δ , the class of closed Finsler n -manifolds (M, F) with injectivity radius $i_M \geq i$, $\Lambda_F \leq \delta$ and $\mu(M) \leq V$, contains at most finitely many homotopy types. Here, $\mu(M)$ is either the Busemann-Hausdorff volume or the Holmes-Thompson volume of M .

Proof. Let c_M denote the contractibility radius of (M, F) (cf. [24]). Since $c_M \geq i_M \geq i$, (M, F) is LGC(ρ), where ρ is the identity map of $[0, i]$. Corollary 5.6 implies that $\mu(B_p^+(\epsilon)) \geq C(n, \delta)\epsilon^n$ for all $p \in M$ and $\epsilon < i/(1 + \sqrt{\delta})$. It follows from [20, Proposition 3.11] that

$$\text{Cov}(M, \epsilon) \leq \frac{\mu(M)}{C(n, \delta)(\epsilon/(2\sqrt{\delta}))^n} = C'(n, \delta, V)\epsilon^{-n}.$$

Define the covering function $N(\epsilon) := C'(n, \delta, V)\epsilon^{-n}$, $\epsilon \in (0, i/(1 + \sqrt{\delta}))$. The conclusion now follows from Lemma 5.8. □

One can easily see that Theorem 5.9 implies Yamaguchi's finiteness theorem [22] and [24, Theorem 1.3].

6. APPENDIX

Proposition 6.1. *Let $(M, F, d\mu)$ be a closed Finsler manifold. Then*

$$\mathbb{I}(M, d\mu) \leq \mathcal{S}(M, d\mu) \leq 2\mathbb{I}(M, d\mu).$$

Proof. Fix Γ with $\mu(M_1) \leq \mu(M_2)$. Define a Lipschitz function f_ϵ^+ by

$$f_\epsilon^+(x) := \begin{cases} 1, & x \in M_1, d(\Gamma, x) \geq \epsilon, \\ \frac{1}{\epsilon}d(\Gamma, x), & x \in M_1, d(\Gamma, x) < \epsilon, \\ 0, & x \in M_2. \end{cases}$$

By letting $\epsilon \rightarrow 0^+$, we obtain that

$$\begin{aligned} \inf_{\alpha \in \mathbb{R}} \left(\int_M |f_\epsilon^+ - \alpha|^{\frac{n}{n-1}} d\mu \right)^{n-1} &\geq \inf_{\alpha \in \mathbb{R}} \{ |1 - \alpha|^{\frac{n}{n-1}} \mu(M_1) + |\alpha|^{\frac{n}{n-1}} \mu(M_2) \}^{n-1} \\ &\geq \mu(M_1)^{n-1} \inf_{\alpha \in \mathbb{R}} \{ |1 - \alpha|^{\frac{n}{n-1}} + |\alpha|^{\frac{n}{n-1}} \}^{n-1} \\ &\geq \mu(M_1)^{n-1} / 2. \end{aligned}$$

Set $\rho_+(x) = d(\Gamma, x)$, $x \in \overline{M_1}$. Lemma 3.2 yields that $\nabla \rho_+|_\Gamma = \mathbf{n}_+$, where \mathbf{n}_+ denotes the unit inward normal vector field along $\partial M_1 = \Gamma$. By the co-area formula (cf. [18, Theorem 3.3.1]), we see that

$$\int_M F^*(df_\epsilon^+) d\mu = \frac{1}{\epsilon} \int_0^\epsilon dt \int_{\rho_+^{-1}(t)} dA_+ \rightarrow A_+(\Gamma).$$

Hence, $2A_+(\Gamma)^n \geq \mathcal{S}(M, d\mu) \cdot \mu(M_1)^{n-1}$. Similarly, define a Lipschitz function f_ϵ^- by

$$f_\epsilon^-(x) := \begin{cases} 0, & x \in M_2, d(\Gamma, x) > \epsilon \\ \frac{1}{\epsilon}d(\Gamma, x) - 1, & x \in M_2, d(\Gamma, x) \leq \epsilon, \\ -1, & x \in M_1. \end{cases}$$

Then one can show $2A_-(\Gamma)^n \geq \mathcal{S}(M, d\mu) \cdot \mu(M_1)^{n-1}$. Therefore, $\mathcal{S}(M, d\mu) \leq 2\mathbb{I}(M, d\mu)$.

Given $f \in C^\infty$, let α_0 be a median of f , i.e.,

$$\mu(\{x : f(x) \geq \alpha_0\}) \geq \frac{1}{2}\mu(M), \quad \mu(\{x : f(x) \leq \alpha_0\}) \geq \frac{1}{2}\mu(M).$$

Set $M_1 := \{x : f(x) < \alpha_0\}$ and $M_2 := \{x : f(x) > \alpha_0\}$. Then $\mu(M_i) \leq \mu(M)/2$, for $i = 1, 2$. Let $h := f - \alpha_0$ and $h_i := h|_{M_i} \in C_c^\infty(M_i)$, $i = 1, 2$.

Let $M_t := \{x : h_2(x) > t\}$. Since $\mu(M_t)$ is decreasing, we have

$$\begin{aligned} \frac{d}{ds} \left(\int_0^s \mu(M_t)^{\frac{n-1}{n}} dt \right)^{\frac{n}{n-1}} &= \frac{n}{n-1} \left(\int_0^s \mu(M_t)^{\frac{n-1}{n}} dt \right)^{\frac{1}{n-1}} \mu(M_s)^{\frac{n-1}{n}} \\ &\geq \frac{n}{n-1} s^{\frac{1}{n-1}} \mu(M_s), \end{aligned}$$

which implies

$$\left(\int_0^s \mu(M_t)^{\frac{n-1}{n}} dt \right)^{\frac{n}{n-1}} \geq \int_0^s \mu(M_t) dt^{\frac{n}{n-1}}.$$

Note that ∇h_2 is the inward normal vector field along ∂M_t . Thus,

$$\begin{aligned} \int_{M_2} F^*(dh_2)d\mu &= \int_0^\infty A_+(\partial M_t)dt \geq \mathbb{I}(M_2)^{\frac{1}{n}} \int_0^\infty \mu(M_t)^{\frac{n-1}{n}} dt \\ &\geq \mathbb{I}(M_2)^{\frac{1}{n}} \left(\int_0^\infty \mu(M_t) dt^{\frac{n}{n-1}} \right)^{\frac{n-1}{n}} = \mathbb{I}(M_2)^{\frac{1}{n}} \left(- \int_0^\infty t^{\frac{n}{n-1}} d\mu(M_t) \right)^{\frac{n-1}{n}} \\ &= \mathbb{I}(M_2)^{\frac{1}{n}} \left(\int_0^\infty t^{\frac{n}{n-1}} dt \int_{\partial M_t} \frac{dA_{\nabla h_2}}{F^*(dh_2)} \right)^{\frac{n-1}{n}} = \mathbb{I}(M_2)^{\frac{1}{n}} \left(\int_M h_2^{\frac{n}{n-1}} d\mu \right)^{\frac{n-1}{n}}. \end{aligned}$$

Here, $\mathbb{I}(M_i)$ is defined by

$$\inf_{\Omega} \frac{\min\{A_{\pm}(\partial\Omega)\}^n}{\mu(\Omega)^{n-1}},$$

where Ω range over all open submanifolds of M_i with compact closures in M_i and smooth boundary. Clearly, $\mathbb{I}(M_i) \neq 0$.

Likewise, one can show that $\int_{M_1} F^*(dh_1)d\mu \geq \mathbb{I}(M_1)^{\frac{1}{n}} \|h_1\|_{n/(n-1)}$. Since $\mu(M_i) \leq \mu(M)/2$, $\mathbb{I}(M_i) \geq \mathbb{I}(M, d\mu)$. Let χ_i be the characteristic function of M_i , $i = 1, 2$. Thus,

$$\begin{aligned} \int_M F^*(df)d\mu &= \int_M F^*(dh)d\mu = \sum_j \int_{M_j} F^*(dh_j)d\mu \\ &\geq \mathbb{I}(M, d\mu)^{\frac{1}{n}} \sum_j \left\{ \int_M \chi_j |f - \alpha_0|^{\frac{n}{n-1}} \right\}^{\frac{n-1}{n}} \\ &\geq \mathbb{I}(M, d\mu)^{\frac{1}{n}} \|f - \alpha_0\|_{\frac{n}{n-1}} \geq \mathbb{I}(M, d\mu)^{\frac{1}{n}} \inf_{\alpha \in \mathbb{R}} \|f - \alpha\|_{\frac{n}{n-1}}, \end{aligned}$$

which implies that $\mathcal{S}(M, d\mu) \geq \mathbb{I}(M, d\mu)$. \square

REFERENCES

- [1] J. Alvarez-Paiva and G. Berck, *What is wrong with the Hausdorff measure in Finsler spaces*, Adv. in Math., **204**(2006), 647-663.
- [2] J. Alvarez-Paiva and A.C. Thompson, *Volumes in normed and Finsler spaces*, A Sampler of Riemann-Finsler geometry (Cambridge) (D. Bao, R. Bryant, S.S. Chern, and Z. Shen, eds.), Cambridge University Press, 2004, pp. 1-49.
- [3] D. Bao, S. S. Chern and Z. Shen, *An introduction to Riemannian-Finsler geometry*, GTM **200**, Springer-Verlag, 2000.
- [4] D. Burago, Y. Burago and S. Ivanov, *A course in metric geometry*, A.M.S., 2001.
- [5] I. Chavel, *Eigenvalues in Riemannian geometry*, Academic Press, New York, 1984.
- [6] B. Chen, *Some geometric and analysis problems in Finsler geometry*, Doctoral thesis, Zhejiang University, 2010.
- [7] C. Croke, *A sharp four dimensional isoperimetric inequality*, Comment. Math. Helv. **59**(1984), 187-192.
- [8] C. Croke, *Curvature Free Volume Estimates*, Inventiones Mathematicae, **76**(1984), 515-521.
- [9] C. Croke, N. Dairbekov, *Lengths and volumes in Riemannian manifolds*, Duke Math. J., **125** (2004), 1-14.
- [10] C. Croke, *Some isoperimetric inequalities and eigenvalue estimates*, Ann. Sci. Ec. Norm. Super, Ser. **13**(1980), 419-435.
- [11] C. Croke, and M. Katz, *Universal volume bounds in Riemannian manifolds*, Surveys in Differential Geometry VIII, Lectures on Geometry and Topology held in honor of Calabi, Lawson, Siu, and Uhlenbeck at Harvard University, May 3-5, 2002, edited by S.T. Yau (Somerville, MA: International Press, 2003.) pp. 109-137.
- [12] D. Egloff, *Uniform Finsler Hadamard manifolds*, Ann. Inst. Henri Poincaré, **66**(1997), 323-357.

- [13] M. Ledoux, *A simple analytic proof of an inequality by P. Buser*, Proc. Amer. Math. Soc. **121**(1994), 951-959.
- [14] Y. Ge and Z. Shen, *Eigenvalues and eigenfunctions of metric measure manifolds*, Proc. London Math. Soc., **82**(2001), 725-746.
- [15] H. Rademacher, *Nonreversible Finsler metrics of positive ag curvature*, A sampler of Riemann-Finsler geometry, Cambridge Univ. Press, Cambridge, 2004, pp. 261-302.
- [16] L. Santaló, *Integral Geometry and Geometric Probability*, Encyclopedia Math. Appl., Addison-Wesley, Reading, MA, 1976.
- [17] L. Santaló, *Measure of sets of geodesics in a Riemannian space and applications to integral formulas in elliptic and hyperbolic spaces*, Summa Brasil. Math., **3**(1952), 1-11.
- [18] Z. Shen, *Lectures on Finsler geometry*, World Sci., Singapore, 2001.
- [19] Z. Shen, *The non-linear Laplacian for Finsler manifolds*, The theory of Finslerian Laplacians and applications, vol. 459 of Math. Appl., Kluwer Acad. Publ., Dordrecht, 1998, pp. 187-198.
- [20] Y. Shen and W. Zhao, *Gromov pre-compactness theorems for nonreversible Finsler manifolds*, Diff. Geom. Appl., **28**(2010), 565-581.
- [21] B. Wu, *Volume form and its applications in Finsler geometry*, Publ. Math. Debrecen, **78**(2011), 723-741.
- [22] T. Yamaguchi, *Homotopy type finiteness theorems for certain precompact families of Riemannian manifolds*, Proc. Am. Math. Soc. **102**(1988), 660-666.
- [23] W. Zhao and Y. Shen, *A Universal Volume Comparison Theorem for Finsler Manifolds and Related Results*, Can. J. Math., **65**(2013), 1401-1435.
- [24] W. Zhao, *Homotopy finiteness theorems for Finsler manifolds*, Publ. Math. Debrecen, **83**(2013), 329-358.

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